

$E_8 \times E_8$ Small Instantons in Matrix Theory

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A formulation of new six-dimensional theories with $(1,0)$ supersymmetry and E_8 global symmetry is proposed. The model is based on the large n theory describing n D-strings interacting with parallel D-fivebranes in Type I string theory.

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1. Introduction

In the past few years nonperturbative effects in string theory have been found that are relevant for arbitrarily small string coupling constant. One class of such effects is the small instanton of the $SO(32)$ heterotic string theory compactified on $K3$, first studied by Witten [1]. He argued that quantum effects lead to an additional $Sp(2)$ gauge symmetry supported at the core of the instanton, together with a set of massless hypermultiplets. The moduli space of the Higgs branch of these new degrees of freedom matches precisely with the $SO(32)$ instanton moduli space obtained via the ADHM construction [2–4].

The $E_8 \times E_8$ small instantons in heterotic string theory on $K3$ have remained somewhat of a mystery. If we consider an instanton in a single E_8 factor, as the instanton shrinks to zero size there is a phase transition from a Higgs branch with 29 massless hypermultiplets, representing the moduli of a finite size instanton of E_8 , to a Coulomb branch with a single massless tensor multiplet. It has been argued the critical point is described by a tensionless string theory in six dimensions [5,6,7]. There is evidence these theories are local quantum field theories at non-trivial renormalization group fixed points [8,6,9].

Further insight may be gained by examining these tensionless strings from the M-theory perspective. M theory compactified on S^1/\mathbb{Z}_2 [10] has been conjectured to give a strong-coupling description of the $E_8 \times E_8$ heterotic string. From this point of view, the tensionless strings discussed above arise from open membranes stretching between a fivebrane and one of the “end of the world” ninebranes of the S^1/\mathbb{Z}_2 compactification [5].

Using the duality between Type IA and M-theory on $S^1 \times S^1/\mathbb{Z}_2$, we can represent the small E_8 instanton by a D-fourbrane of Type IA approaching an orientifold plane. The full E_8 symmetry will be recovered in the infinite-coupling limit when 8 D-eightbranes also lie on the orientifold plane [11]. The worldvolume theory on the fourbrane will then become one of the five-dimensional supersymmetric non-trivial interacting fixed-point field theories studied by Seiberg [11].¹

A quantum formulation of M-theory on S^1/\mathbb{Z}_2 has been recently proposed in terms of the matrix mechanics of a system of Type IA D-particles [18–20]. The perturbative heterotic string may be directly recovered from this formulation [21,22] and the leading order string interactions are reproduced [22]. Related results have also been obtained in [23].

¹ The duality between the $E_8 \times E_8$ heterotic string on $K3$ and F theory [12] on an elliptic Calabi-Yau threefold, offers yet another means of studying the tensionless string theories that describe small E_8 instantons. The small instanton limit corresponds to the collapse of a del Pezzo surface in the threefold [13–17].

In this paper we use this heterotic matrix theory to propose for the first time a nonperturbative formulation of these tensionless string theories. In Type IB language, we can realize these theories as a collection of D-strings, D-fivebranes, and D-ninebranes, where we take n , the number of D-strings to infinity. This configuration is of course T-dual to the Type IA configuration discussed above. By keeping the radius of the S^1 that the D-strings wrap finite, we may also use this model to formulate new six-dimensional non-critical string theories which flow to the tensionless string theories in the limit that the radius vanishes. Such non-critical string theories have previously been considered in the Type II context in [24]. In section 2 we set up the Lagrangian describing this configuration of D-branes. In section 3 we show how the resulting large- n gauged linear sigma model is obtained from the original heterotic matrix model. For finite n these results may be interpreted in terms of a discretized light-cone quantization of M-theory with the longitudinal direction compactified on a circle of finite radius with a Wilson line. Section 4 is devoted to a study of the spectrum of BPS states of this theory. The Calabi-Yau approach [14,15] predicts a tower of states with E_8 quantum numbers. When the appropriate Wilson line is introduced, these decompose into representations of $SO(16)$. These results are shown to be consistent with the Matrix description of the theory.

While this work was in an advanced stage, related work proposing a large n formulation of six-dimensional theories with $(2,0)$ supersymmetry appeared [25,26]. In the context of theories with $(1,0)$ supersymmetry, related ideas have also been considered in [27].

2. D-strings and D-fivebranes in Type IB

The Lagrangian on the worldsheet of a single D-string probe in a background of k D-fivebranes and 32 D-ninebranes was found by Douglas [28,29] to be equivalent to a linear sigma model studied earlier by Witten [30] in the context of the ADHM construction. Here we will need the generalization of this to n D-strings, which introduces an additional $O(n)$ gauge symmetry. We will follow the notation of [29], adding $O(n)$ indices as needed.

The Lorentz group decomposes into $SO(1,1) \times SO(4)_I \times SO(4)_E$, where $SO(4)_I$ corresponds to rotations within the fivebranes, while $SO(4)_E$ corresponds to rotations in the directions transverse to the fivebranes. Each $SO(4)$ decomposes in turn into a product of two $SU(2)$ factors. Doublets in the two $SU(2)$'s of $SO(4)_E$ are labeled by indices A and Y , while those of $SO(4)_I$ are labeled by A' and \tilde{A}' .

The fields on the worldsheet of the D-strings transform under a $Sp(2k) \times SO(32)$ global symmetry group. The index m will label the fundamental of $Sp(2k)$ and M will label the fundamental of $SO(32)$.

The worldsheet fields are as follows

boson	fermion	$O(n)$ rep
A_μ	$\psi_+^{AA'}, \psi_+^{\tilde{A}'Y}$	adjoint
b^{AY}	$\psi_-^{A'Y}$	symmetric
$b^{A'\tilde{A}'}$	$\psi_-^{A\tilde{A}'}$	symmetric
$\phi^{A'm}$	χ_-^{Am}	fundamental
	χ_+^{Ym}	fundamental
	λ_+^M	fundamental.

(2.1)

Here A_μ is an $O(n)$ gauge field. The λ_+^M are real. All the scalars and their superpartners satisfy a reality condition of the form

$$b^{AY} = \epsilon^{AB} \epsilon^{YZ} \bar{b}_{BZ} . \quad (2.2)$$

ϵ is replaced by the $Sp(2k)$ invariant antisymmetric tensor when appropriate. All indices are raised and lowered as $v_A = \epsilon_{AB} v^B$. The worldsheet coordinates are denoted by σ and τ , with worldsheet metric $ds^2 = d\tau^2 - d\sigma^2$, and we define light-cone coordinates $\sigma^\pm = (\tau \pm \sigma)/\sqrt{2}$.

The configuration of D-strings and D-fivebranes is invariant under $(0, 4)$ supersymmetry, i.e. there are four real right-moving supercharges. These supercharges satisfy a reality condition of the form

$$Q^{AA'} = \epsilon^{AB} \epsilon^{A'B'} Q_{BB'}^\dagger . \quad (2.3)$$

The R-symmetry group is $SO(4) = SU(2) \times SU(2)$. We will refer to the two different $SU(2)$ factors as F and F' . In the infrared the model will flow to a theory with $\mathcal{N} = 4$ superconformal invariance which is only invariant under a single $SU(2)$. We will construct the model so that invariance under F' is manifest. The supersymmetry algebra is taken to be

$$\{Q^{AA'}, Q^{BB'}\} = \epsilon^{AB} \epsilon^{A'B'} P^+ , \quad (2.4)$$

where $P^+ = -i\partial/\partial\sigma^-$.

Let us now consider the different supermultiplets that appear in the model. We will use $\eta_+^{AA'}$ to parameterize the $(0, 4)$ supersymmetry transformations. The b^{AY} and $\psi_-^{A'Y}$ form a standard multiplet which transforms as

$$\delta b^{AY} = i\epsilon_{A'B'} \eta_+^{AA'} \psi_-^{B'Y} , \quad \delta \psi_-^{A'Y} = \epsilon_{AB} \eta_+^{AA'} D_- b^{BY} . \quad (2.5)$$

Here we define the covariant derivative by $D_\mu = \partial_\mu - gA_\mu^\alpha T_R^\alpha$, where T_R^α is the gauge generator for the representation R . The $b^{A'\tilde{A}'}$ and $\psi_-^{A\tilde{A}'}$ form a twisted multiplet which transforms as

$$\delta b^{A'\tilde{A}'} = i\epsilon_{AB}\eta_+^{AA'}\psi_-^{B\tilde{A}'}, \quad \delta\psi_-^{A\tilde{A}'} = \epsilon_{A'B'}\eta_+^{AA'}D_-b^{B'\tilde{A}'} . \quad (2.6)$$

The fields $\phi^{A'm}$ and χ_-^{Am} also form a twisted multiplet and transform as (2.6).

The gauge multiplet consists of A_μ , $\psi_+^{AA'}$ and $\psi_+^{\tilde{A}'Y}$. We will write A_μ in light-cone coordinates in terms of components A_+ and A_- . The terms in the Lagrangian and the supersymmetry transformations involving these fields will be determined by the Noether procedure. Namely, we begin with an action with just a $U(1)$ gauge symmetry and an $O(n)$ global symmetry and then add terms to the Lagrangian and supersymmetry transformations order-by-order in the gauge coupling constant g to obtain a theory with $(0,4)$ supersymmetry and $O(n)$ gauge invariance. One obtains the following result for the supersymmetry transformations

$$\begin{aligned} \delta A_+ &= -i\epsilon_{AB}\epsilon_{A'B'}\eta_+^{AA'}\psi_+^{BB'} \\ \delta A_- &= 0 \\ \delta\psi_+^{AA'} &= F_{+=}\eta_+^{AA'} + gD_{BB'}^{AA'}\eta_+^{BB'} \\ \delta\psi_+^{\tilde{A}'Y} &= gD_{BB'}^{\tilde{A}'Y}\eta_+^{BB'} , \end{aligned} \quad (2.7)$$

where we have defined

$$\begin{aligned} D_{BB'}^{AA'} &= b^{AY}T_S b_{BY}\delta_B^{A'} + b^{A'\tilde{A}'}T_S b_{B'\tilde{A}'}\delta_B^A + \phi^{A'm}T_F\phi_{B'm}\delta_B^A \\ D_{BB'}^{\tilde{A}'Y} &= 2b_A^YT_S b_{B'}^{\tilde{A}'} , \end{aligned} \quad (2.8)$$

and $S(F)$ refers to the symmetric (fundamental) rep. The field strength is defined as $F_{+=} = \partial_+A_- - \partial_-A_+ - g[A_-, A_+]$.

The purely left-moving fermionic multiplets are χ_+^{Ym} and λ_+^M , and we will frequently denote these as $\lambda_+^a = (\chi_+^{Ym}, \lambda_+^M)$. The supersymmetry transformations of these fields are determined in terms of the functions $C_{AA'}^a$ of the bosonic fields as

$$\delta\lambda_+^a = \eta_+^{AA'}C_{AA'}^a . \quad (2.9)$$

The functions $C_{AA'}^a$ have global $O(n)$ symmetry. Since the terms in the Lagrangian dependent on the $C_{AA'}^a$ do not contain derivatives, they do not alter the supersymmetry transformations (2.7). The functions $C_{AA'}^a$ may be determined by demanding $(0,4)$ supersymmetry [30]. We require that up to gauge transformations

$$(\delta_{\eta'}\delta_\eta - \delta_\eta\delta_{\eta'})\lambda_+^a = -i\epsilon_{A'B'}\epsilon_{AB}\eta_+^{AA'}\eta_+^{BB'}D_-\lambda_+^a = -i\epsilon_{A'B'}\epsilon_{AB}\eta_+^{AA'}\eta_+^{BB'}G_\theta^a\rho^\theta , \quad (2.10)$$

where ρ^θ includes all the right-movers, and G_θ^a is a function of the bosonic scalars. The covariant derivative $D_\mu = \partial_\mu - gA_\mu$ when acting on fields in the fundamental representation. Equation (2.10) implies the condition

$$0 = \frac{\partial C_{AA'}^a}{\partial b^{BY}} + \frac{\partial C_{BA'}^a}{\partial b^{AY}} = \frac{\partial C_{AA'}^a}{\partial \phi^{B'm}} + \frac{\partial C_{AB'}^a}{\partial \phi^{A'm}} = \frac{\partial C_{AA'}^a}{\partial b^{B'\tilde{B}'}} + \frac{\partial C_{AB'}^a}{\partial b^{A'\tilde{B}'}} . \quad (2.11)$$

The part of the Lagrangian containing λ_+ is then determined to be

$$\int d^2\sigma \left(\frac{i}{2} \lambda_+^a D_- \lambda_+^a - \frac{i}{2} \lambda_+^a G_{a\theta} \rho_-^\theta \right) , \quad (2.12)$$

with

$$G_{\theta}^a \rho_-^\theta = \frac{1}{2} \left(\epsilon^{BD} \frac{\partial C_{BB'}^a}{\partial b^{DY}} \psi_-^{B'Y} + \epsilon^{B'D'} \frac{\partial C_{BB'}^a}{\partial \phi^{D'Y'}} \chi_-^{BY'} + \epsilon^{B'D'} \frac{\partial C_{BB'}^a}{\partial b^{D'\tilde{D}'}} \psi_-^{B\tilde{D}'} \right) . \quad (2.13)$$

To obtain a Lagrangian with the full $(0, 4)$ supersymmetry it is useful to compare to the case with $(0, 1)$ supersymmetry where a superfield formalism is available. We have

$$\delta \lambda_+^a = \eta F^a , \quad (2.14)$$

where F is an auxiliary field. The potential energy of the system is $\sum_a (F^a)^2/2$. On shell F^a should be identified with C^a , say as $F^a = c^{AA'} C_{AA'}^a$, so to obtain a theory with $(0, 4)$ supersymmetry we must require that V be independent of $c^{AA'}$. Note c should be normalized so that $c^2 = 1$. This leads to the condition

$$0 = \sum_a \text{Tr}(C_{AA'}^a C_{BB'}^a + C_{BA'}^a C_{AB'}^a) . \quad (2.15)$$

Solving conditions (2.15) and (2.11) yields

$$C_{AA'}^M = g h_{Am}^M \phi_{A'm} , \quad C_{AA'}^{Ym} = g \phi_{A'n} (X_{mn}^{AY} - b^{AY} \delta_{mn}) . \quad (2.16)$$

Here h and X are fields in the D-fivebrane worldvolume theory, which appear as background fields from the point of view of the D-strings. These fields must satisfy the D-flatness conditions of the fivebrane worldvolume theory,

$$h_m^{M(A} h_{Mn}^{B)} + \epsilon^{pq} \epsilon_{YZ} X_{mp}^{(AY} X_{nq}^{B)Z} = 0 . \quad (2.17)$$

The final Lagrangian is given by

$$\begin{aligned}
L = & \frac{1}{2} \int d^2\sigma \text{Tr} \left(D_{\pm} b^{AY} D_{\pm} b_{AY} + D_{\pm} b^{A'\tilde{A}'} D_{\pm} b_{A'\tilde{A}'} + D_{\pm} \phi^{A'm} D_{\pm} \phi_{A'm} \right. \\
& \left. - i\rho_{-}^{\theta} D_{\pm} \rho_{-\theta} + i\lambda_{+}^a D_{\pm} \lambda_{+a} - F_{\pm=}^2 + i\psi_{+} D_{\pm} \psi_{+} \right) \\
& - \frac{i}{4} \int d^2\sigma \text{Tr} \left(\lambda_{+a} \left(\epsilon^{BD} \frac{\partial C_{BB'}^a}{\partial b^{DY}} \psi_{-}^{B'Y} + \epsilon^{B'D'} \frac{\partial C_{BB'}^a}{\partial \phi^{D'm}} \chi_{-}^{Bm} \right) \right. \\
& \left. - \frac{1}{8} \int d^2\sigma \text{Tr} \left(\epsilon^{AB} \epsilon^{A'B'} C_{aAA'} C_{BB'}^a \right) \right. \\
& \left. - ig \int d^2\sigma \text{Tr} \left(b_{AY} \psi_{+}^{AA'} \psi_{-A'}^Y + b_{A'\tilde{A}'} \psi_{+}^{AA'} \psi_{-\tilde{A}'}^A + \phi_{A'm} \psi_{+}^{AA'} \chi_{-A}^m \right. \right. \\
& \left. \left. + b_{AY} \psi_{+}^{\tilde{A}'Y} \psi_{-\tilde{A}'}^A + b_{A'\tilde{A}'} \psi_{+}^{\tilde{A}'Y} \psi_{-Y}^{A'} \right) \right. \\
& \left. - \frac{g^2}{4} \int d^2\sigma \text{Tr} \epsilon^{BC} \epsilon^{B'C'} \left(\epsilon_{AD} \epsilon_{A'D'} D_{BB'}^{AA'} D_{CC'}^{DD'} + \epsilon_{\tilde{A}\tilde{B}'} \epsilon_{Y Z} D_{BB'}^{\tilde{A}'Y} D_{CC'}^{\tilde{B}'Z} \right) \right). \tag{2.18}
\end{aligned}$$

When the number of D-fivebranes k vanishes, and one introduces an $SO(32)$ Wilson line, which makes half of the λ_{+}^M periodic and the other half anti-periodic, this reduces to the two-dimensional $(0, 8)$ system considered in [22]. When the number of D-strings is one, this reduces to the linear sigma model considered in [30,29].

As discussed in the introduction, the small instanton limit corresponds to taking the Type IA coupling constant to infinity. This translates into taking the coupling g of (2.18) to infinity, which in turn corresponds to taking the infrared limit of the theory. To describe the uncompactified case the size of the σ direction should be sent to zero. When combined with taking the large n limit and setting X and h to zero, this gives a matrix description of small instantons in $E_8 \times E_8$ heterotic string theory.

For finite n we will argue below this gives a description of $E_8 \times E_8$ small instantons, or equivalently gauge fivebranes of heterotic string theory, in which the longitudinal direction parallel to the fivebranes has been compactified. Taking n finite then corresponds to considering the discretized light-cone quantization of this theory. We will elaborate on the details of this further in the following sections.

Note that although the X and h couplings vanish for the case of small instantons, retaining these couplings will clarify the infrared behavior of the theory, as we will see later. The X fields describe the positions of the D-fivebranes in the four transverse dimensions. The additional degrees of freedom that describe the positions of the M-theory fivebranes in the eleventh dimension arise when one allows for nontrivial boundary conditions for the $\phi^{A'm}$ fields as one goes around σ . These are the remnants of the Wilson line degrees of freedom of the $Sp(2k)$ gauge group on the worldvolume of the fivebranes.

If the radius of the σ direction is held fixed in the infrared limit, the theory (2.18) gives a matrix formulation of a new noncritical six-dimensional string theory with $E_8 \times E_8$ global symmetry and $(1, 0)$ supersymmetry, analogous to the $(2, 0)$ six-dimensional noncritical string theories recently considered in [24].

3. Matrix Theory Approach

Let us see how the gauged linear sigma model found in the previous section may be derived from the Matrix theory approach. Our starting point is the heterotic Matrix model [18,19,20]. Compactifying on a circle one obtains a two-dimensional $O(n)$ gauge theory as described in [21,22]. Compactifying further on a T^4 one obtains a six-dimensional noncritical string theory discussed in [24,31] compactified on $K3 \times S^1$. The moduli space of vacua in this theory will be $SO(20, 4, \mathbb{Z}) \backslash SO(20, 4) / (SO(20) \times SO(4))$, and the duality group (including mirror symmetry) is identified with $SO(20, 4, \mathbb{Z})$. We work at a point where the $K3$ is T^4/\mathbb{Z}_2 where the T^4 has sizes Σ_i ($i = 1, \dots, 4$) and the S^1 has size Σ_5 , with all angles right angles and the three form set to zero. The parameters of the compactified $(2, 0)$ theory are identified with parameters of the M-theory on a dual $T^4/\mathbb{Z}_2 \times S^1$ (with sizes L_i with $i = 1, \dots, 5$) via

$$\begin{aligned}\Sigma_i &= \frac{l_p^3}{RL_i}, \quad (i = 1, \dots, 4) \\ \Sigma_5 &= \frac{l_p^6}{RL_1 L_2 L_3 L_4} \\ M_s^2 &= \frac{R^2 L_1 L_2 L_3 L_4 L_5}{l_p^9},\end{aligned}\tag{3.1}$$

where R is the length of the longitudinal direction, l_p is the eleven dimensional Planck length, and M_s is string scale of the noncritical string theory.

When $M_s \rightarrow \infty$ this reduces to the $(2, 0)$ field theory (see [32] for a review) compactified on $K3 \times S^1$, as studied in [33]. This should correspond to the heterotic matrix model compactified on T^3 .

In general, this noncritical string theory only looks like a supersymmetric field theory in special corners of the moduli space. Actually, we will chiefly be interested in the limit when the T^4 is very large (i.e. L_1, \dots, L_4 large) with the eleven dimensional Planck length fixed which precisely corresponds to one of these corners. At the point with $SU(2)^{16}$ enhanced gauge symmetry this theory flows in the infrared to a $U(n)$ Yang-Mills theory on a dual orbifold space $S^1 \times T^4/\mathbb{Z}_2$. To relate this to the heterotic Matrix model we need

to consider a limit in which one direction of the T^4 is much smaller than the other length scales. After Kaluza-Klein reduction one obtains a Yang-Mills theory on $S^1 \times T^3/\mathbb{Z}_2$, with certain boundary conditions on the fields:

$$\begin{aligned} A_{0,1}(\sigma, \sigma^i) &= -A_{0,1}^\dagger(\sigma, -\sigma^i) \\ A_a(\sigma, \sigma^i) &= A_a^\dagger(\sigma, -\sigma^i) \\ X_k(\sigma, \sigma^i) &= X_k^\dagger(\sigma, -\sigma^i) , \end{aligned} \tag{3.2}$$

together with extra 1+1-dimensional fermionic degrees of freedom that live at the fixed points [33,34]. We wish to introduce an instanton background into this theory and then integrate out the heavy modes to obtain the effective action. To do this we decompose $U(n) = U(n_0) \times U(n_1)$ and embed the instanton in the $U(n_0)$ factor. The scalars decompose as

$$X_k = \begin{pmatrix} Z_k & Y_k \\ Y_k^\dagger & x_k \end{pmatrix} , \tag{3.3}$$

and a similar equation holds for the fermions. Here Z_k is a $n_0 \times n_0$ matrix, x_k is $n_1 \times n_1$, etc. The leading contribution to the action comes from the zero modes of the Y field (and its fermionic partner) in the background of the instanton. For a k instanton configuration there will be kn_1 such zero modes, which will transform in a bifundamental representation (k, n_1) . In the limit that the size of the dual torus T^3 shrinks to zero size we obtain a description of a small instanton in flat space, and the matrix description reduces to a two-dimensional field theory. For $k = 0$ one finds the usual fields of the heterotic Matrix theory [22]. For $k \neq 0$, when one takes into account the additional projection (3.2) one finds precisely the additional worldsheet fields $\phi^{A'm}$ together with the fermionic fields χ_+^{Ym} and χ_-^{Am} of the previous section. These pick up an additional sign under the \mathbb{Z}_2 projection with respect to the b^{AY} scalars, so the new zero modes have an $Sp(2k)$ global symmetry, rather than $O(2k)$. Gauge symmetry and $(0,4)$ supersymmetry then fixes the effective action to be of the form (2.18).

3.1. Discrete Light-Cone Quantization

Now let us make a few comments on the interpretation of this Lagrangian for finite n . The comments we make here are also relevant for the theory without the instanton background. In Type II case it has been conjectured the finite n Matrix theory describes a DLCQ sector of M theory with the longitudinal direction x^- compactified on a circle of radius R . The same argument can be made in this case. A new feature now is the presence of Wilson line degrees of freedom corresponding to the boundary conditions on the gauge

fermions. In general, these Wilson lines are unconstrained. If however we want to decompactify, by taking a large n limit, the Wilson line must be carefully chosen if we are to obtain a theory with a conventional local spacetime interpretation and $9 + 1$ -dimensional Lorentz invariance. Apparently, the only consistent way to do this is to take the Wilson line as in [22] which gives half of the λ_+^M fields periodic boundary conditions, and the other half antiperiodic boundary conditions as one goes around the S^1 . The absence of manifest $E_8 \times E_8$ invariance is then simply due to the presence of this Wilson line at finite radius. Similar comments have also recently appeared in [35].

4. Spectrum of States

As discussed above, the theory (2.18) for finite n can be used to describe small instantons compactified on a longitudinal circle in the presence of a Wilson line. The large n limit of the theory is conjectured to describe the full uncompactified six-dimensional theory associated with small $E_8 \times E_8$ instantons. In the following we will consider the case of small instantons in a single E_8 factor. This theory may alternatively be studied by considering F-theory compactified on an elliptic Calabi-Yau threefold with a collapsing del Pezzo surface. Compactifying further on a circle yields M-theory on a similar Calabi-Yau, giving a description of the $4+1$ -dimensional theory. The particle-like BPS states that appear from this point of view were studied in [14,15] by counting cycles in Calabi-Yau manifolds.

These results can be summarized in terms of a non-critical string theory with E_8 quantum numbers. For the case of single winding number of this non-critical string, they find the result that the degeneracy $d(n_E)$ of states with momentum n_E is given by

$$q^{-\frac{1}{2}} \sum_{n_E=0}^{\infty} d(n_E) q^{n_E} = \frac{\theta_{E_8}(q)}{\eta(q)^{12}}. \quad (4.1)$$

The $SO(4)_I = SU(2) \times SU(2)$ spacetime quantum numbers of these states may also be deduced. For $n_E = 0$ they find a hypermultiplet singlet of E_8 . The hypermultiplet has spacetime quantum numbers $4(0, 0) \oplus (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The states with $n_E = 1$ and their CPT conjugates form a hypermultiplet in the **248** together with a singlet in the $4(\frac{1}{2}, \frac{1}{2}) \oplus (1, \frac{1}{2}) \oplus (\frac{1}{2}, 1) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$. The states were found by considering the limit in which the del Pezzo surface collapsed to zero size. Precisely at the critical point it is possible for the states with $n_E > 1$ to decay, for example to states with winding number ± 1 and $n_E = 0, \pm 1$.

To compare directly with the Matrix theory results for finite n it is necessary to introduce a Wilson line around the compact direction which breaks the E_8 symmetry to $SO(16)$. This splits the **248** of E_8 into the **120** of $SO(16)$ and the **128**. The masses of these states change in the usual way

$$M = \left| \frac{n_E + A_E P_E}{2R} \right|, \quad (4.2)$$

where A_E is the Wilson line parameter in E_8 variables, and P_E is a point on the E_8 weight lattice.

In general, the relations between the E_8 quantities and the $SO(32)$ quantities that appear in the Matrix theory are

$$n = 2n_E + 2A_E P_E + A_E^2 m_E, \quad 2m + 2AP + A^2 n = m_E, \quad (4.3)$$

where (n, m) are the winding number and momentum of a fundamental $SO(32)$ string, A is the Wilson line $(\frac{1}{2}^8, 0^8)$ in the Cartan subalgebra of $SO(32)$, and P is a point on the lattice $Spin(32)/\mathbb{Z}_2$. Here m_E is the winding number of a fundamental $E_8 \times E_8$ string, which is to be identified with the worldsheet momentum of (2.18). At present we consider the limit that the radius of the σ direction vanishes, so we set $m_E = 0$.

The formula (4.3) implies the singlet with $n_E = 0$ lies at $n = 0$. The **120** of $SO(16)$ at $n_E = 1$ will lie at $n = 2$ with $2(0, 0) \oplus (\frac{1}{2}, 0)$ $SO(4)_I$ quantum numbers. The singlet at $n_E = 1$ will lie at $n = 2$ with $2(\frac{1}{2}, \frac{1}{2}) \oplus (1, \frac{1}{2}) \oplus (0, \frac{1}{2})$ quantum numbers. The **128** at $n_E = 1$ will lie at $n = 1$ with $2(0, 0) \oplus (\frac{1}{2}, 0)$ $SO(4)_I$ quantum numbers. The higher representations of E_8 found in (4.1) will begin to appear at $n = 2$.

The Kaluza-Klein compactification of the six-dimensional field theoretic states gives rise to additional states in five dimensions, which are independent of the wrapped tensionless string states.² The tensor multiplet reduces to a tower of states with $(1, 0) \oplus (0, 0) \oplus 2(\frac{1}{2}, 0)$ $SO(4)_I$ quantum numbers for all n_E , which translates to all even n . The 29 massless hypermultiplets of the six-dimensional theory transform as the $\frac{1}{2}\mathbf{56} + \mathbf{1}$ of the unbroken E_7 of the Higgs branch. When we compactify and turn on Wilson lines, the E_8 is broken to $SO(16)$. The $SO(16)$ is further broken to $SO(12) \times SU(2)$ by the finite size instanton. Likewise the E_7 symmetry of six dimensions will be broken to $SO(12) \times SU(2)$ by the Wilson lines, under which $\mathbf{56} \rightarrow (\mathbf{12}, \mathbf{2}) \oplus (\mathbf{32}, \mathbf{1})$. At the point where the instanton

² In [15] it was noted this spectrum depends on the details of the Calabi-Yau. The Matrix theory with $k = 1$ should correspond to the critical point in the transition between the Higgs branch with 29 massless hypermultiplets and the Coulomb branch with a single tensor multiplet.

shrinks to zero size, these representations will be enlarged to representations of $SO(16)$. The **32** will be enlarged to a spinor **128** which will generate a tower of Kaluza-Klein states with n odd, and $2(0,0) \oplus (\frac{1}{2},0)$ $SO(4)_I$ quantum numbers. Finally the singlet and the **(12, 2)** states will arise from the **120** of $SO(16)$, which will generate a tower of states with n even, and $2(0,0) \oplus (\frac{1}{2},0)$ quantum numbers.

4.1. $n = 1, k = 1$

We first examine the instanton number one case, and set n to one. To deduce the BPS states from the Matrix approach, we consider the conformal field theory that (2.18) flows to in the infrared. The potential energy appearing in (2.18) takes the form

$$V = \sum |C|^2 \sim \phi^2(X^2 + b^2) . \quad (4.4)$$

For generic ADHM data (i.e. D-fivebrane worldvolume fields X and h) there is one branch of the moduli space along which ϕ is massive. In the infrared the Lagrangian will flow to one of the wormhole conformal field theories studied in [36,37]. These theories have $(0,4)$ worldsheet supersymmetry. Naively one might think these would be symmetric under a $SO(4) = SU(2) \times SU(2)$ R-symmetry group, but in fact the $\mathcal{N} = 4$ superconformal algebra is only symmetric under a $SU(2)$ subgroup. On the branch where ϕ is massive the $SU(2)$ that appears in the superconformal algebra is $SU(2)_{A'}$, and $SU(2)_A$ is spontaneously broken.

When $X = 0$ the gauged linear sigma model describes an instanton of zero size. At this point, a second branch of the moduli space appears, along which b becomes massive and ϕ takes on a nonzero expectation value. In this case, the linear sigma model flows to a different superconformal field theory in the infrared where now $SU(2)_A$ appears in the superconformal algebra. Once again one may argue the conformal field theory that appears is one of the wormhole conformal field theories [36,37].

As discussed in [38] a potential contradiction arises in this picture. Naively the two branches meet at $b = \phi = 0$ which would be inconsistent with the origin of the R symmetry of the superconformal algebra as discussed above. This problem is resolved if in the conformal field theory limit the distance to the $b = \phi = 0$ point moves off to infinity allowing the two branches to remain separate.

Let us consider further the conformal field theory that appears in the infrared. One way to study this theory is simply to solve the low-energy spacetime equations of motion order by order in α' . To leading order this yields

$$ds^2 = (db)^2 (e^{2\phi_0} + 8\alpha' \frac{(b^2 + 2X^2)}{(b^2 + X^2)^2} + O(\alpha'^2)) , \quad (4.5)$$

where ϕ_0 is the value of the dilaton at infinity and $(db)^2$ is the usual flat space metric. As $X \rightarrow 0$ the spacetime develops a long tube, consistent with $b = \phi = 0$ being at infinite distance in the conformal field theory. It is possible to show this solution may be corrected order by order in α' to yield an exact conformal field theory [39]. Unfortunately little is known about this exact conformal field theory. The next to leading order solution for the spacetime fields obtained from the ADHM sigma model has been obtained in [40], where it is shown the solution differs from the instanton solution of [36]. Nevertheless the corrections remain consistent with the $b = \phi = 0$ point being at infinite distance.

The construction of the states in the exact conformal field theory is a difficult problem. One approach would be to do a Witten index calculation, however because the ground states will appear as bound states at threshold, this index calculation is rather subtle. Instead we will show the states expected arise in two different limits where we may use algebraic CFT techniques to analyze the theory.

The first limit we consider is the CFT describing the region far from the worm-hole, which we may think of as the Coulomb branch of the two-dimensional theory. This approaches flat space, so the CFT is the usual free theory heterotic theory with $(0, 8)$ supersymmetry. The spectrum of states is the same as in [22] so for $n = 1$ one finds the **128** of $SO(16)$. This lies in the $\mathbf{8}_V \oplus \mathbf{8}_S$ of $SO(8)$ which decomposes into $(\frac{1}{2}, \frac{1}{2}) \oplus 4(0, 0) \oplus 2(\frac{1}{2}, 0) \oplus 2(0, \frac{1}{2})$ under $SO(4)_I$. These are the usual states of the compactified ten-dimensional heterotic string theory and are expected to decouple from the intrinsically six-dimensional degrees of freedom associated with the small instanton.

The second limit we consider, where we may still explicitly describe the CFT is the long tube region of the spacetime that appears in the zero-size limit $X \rightarrow 0$. This can be described by a tractable conformal field theory which is a supersymmetric $SU(2)$ WZW model together with a free field with a linear dilaton [36], and the usual free left-moving gauge fermions λ_+^M . The level k_w of the WZW model is identified with the charge of the solution. We take $k_w = 1$ in the case at hand. With an appropriately defined energy momentum tensor, the central charge of this theory is 6 [41]. The mass-shell condition is

$$m^2 = -1 + \frac{1}{2}P_L^2 + \frac{1}{2}P_R^2 + \frac{c_R + c_{\bar{R}}}{2 + k_w} + N + \bar{N} , \quad (4.6)$$

where P_L and P_R are the left and right-moving momenta, c_R is the second Casimir of the representation R , and N and \bar{N} are left and right-moving oscillator numbers. The level-matching condition becomes

$$\frac{1}{2}P_R^2 + \frac{c_{\bar{R}}}{2 + k_w} + \bar{N} = \frac{1}{2}P_L^2 + \frac{c_R}{2 + k_w} + N - 1 . \quad (4.7)$$

The BPS condition requires the right-movers to be in their ground states. The states found from the Calabi-Yau point of view are obtained by taking R and \bar{R} to be singlets, and the derivation of the spectrum proceeds as in the $(0, 8)$ case, with identical results.

Now let us consider the Higgs branch of the two-dimensional theory where ϕ has a non-zero vev. The CFT that appears on this branch is similar to the one just considered, but now orbifolded by the \mathbb{Z}_2 symmetry that acts on ϕ and χ . The quantum numbers of the ground states arise from the quantization of the fermion zero modes λ_+^M and χ_-^{Am} . The $Sp(2)$ quantum numbers are associated with the tensionless string winding number. One finds the **128** of $SO(16)$ in the $2(0, 0) \oplus (\frac{1}{2}, 0)$ of $SO(4)_I$ and in the $2(0, 0) \oplus (\frac{1}{2}, 0)$ of $Sp(2) \times SU(2)_A$, which contains the states expected from the Kaluza-Klein compactification of the six-dimensional hypermultiplets, and the wrapped tensionless strings.

We see therefore that in either limit, the conformal field theories contain all the states predicted from the Calabi-Yau approach. Of course, we are really interested in normalizable states of the exact conformal field theory which interpolates between these two limits. It is expected the additional states we have found will turn out to be non-normalizable when one constructs their wavefunctions globally. It would be very interesting to do a Witten index calculation to confirm this.

4.2. $k = 1, n > 1$

Now we consider the case where we have multiple D-strings and a single D-fivebrane. Naively one might think the picture would be similar to [22], namely that in the infrared the theory would flow to a symmetrized product of the conformal field theories discussed in the previous subsection. The difference here is that now there exist marginal deformations of the CFT consistent with the $SO(4)_I$ rotational invariance and gauge symmetry. Following [25] where the fivebrane of Type II was considered, we may argue there is a unique marginal operator which preserves the $SO(4)_E$ group of rotations transverse to the fivebrane. In [25] the coupling of this marginal operator was fixed by demanding that a worldsheet theta angle vanish. Presumably a similar argument will apply here. The undeformed orbifold conformal field theory leads to a non-vanishing value for the worldsheet theta angle, so does not describe the correct infrared fixed point for the gauge theory.

If one is only interested in the spectrum of BPS states, one could argue these are insensitive to the deformation, allowing computations to be performed at the orbifold point. It is unlikely one can trust this argument in detail for the present case. However let us follow this logic for the case $n = 2$ and apply a Born-Oppenheimer approximation as in the previous subsection. For the singlet and **120** representations of $SO(16)$ one does indeed find results consistent with states expected from Kaluza-Klein compactification of

the six-dimensional field theory, together with wrapped tensionless string states predicted by the Calabi-Yau approach. However, there is no sign of the higher representations of E_8 predicted by [15,14]. It is possible these arise as bound states at threshold once the interactions induced by the marginal operator are properly included, or alternatively they may only be stable on the Coulomb branch of the six-dimensional theory. Similar arguments can be made for the $n > 2$ case.

4.3. $k > 1$

For many D-fivebranes, and a single D-string (2.18) will flow in the infrared to a wormhole conformal field theory [36], with level number $k_w = k$. The construction of the BPS states proceeds as above for the case $n = 1$. For multiple D-strings one will again find a symmetric product of these conformal field theories, up to marginal deformations involving twist fields. Further consideration of this case is beyond the scope of the present work.

4.4. Coulomb Branch

The six-dimensional E_8 theory also has a Coulomb branch where the massless states are a single tensor multiplet. To see this branch from the Matrix point of view we must set $X = h = 0$ and turn on the Wilson line degrees of freedom discussed at the end of section two. These are identified with the position of the fivebrane relative to the end of the world in the eleventh direction. The Lagrangian flows to that of a Type IIB theory with a D-fivebrane interacting with $n/2$ D-strings (here we must set n even). The $O(n)$ gauge symmetry is broken to $U(n/2)$. For $n = 2$ this configuration is T-dual to a D-particle plus D-fourbrane system in Type IIA. In [42] this was shown to yield a single bound state with $\mathcal{N} = 4$ vector multiplet quantum numbers from the four-dimensional point of view. The $\mathcal{N} = 4$ vector multiplet decomposes into a $\mathcal{N} = 2$ vector multiplet and a hypermultiplet. The $\mathcal{N} = 2$ vector is precisely what is expected by compactifying the six-dimensional tensor multiplet. The hypermultiplet represents the center of mass degree of freedom of the fivebrane which decouples from the six-dimensional theory.

For $n > 2$, the theory will flow to a $(4, 4)$ conformal field theory of the type considered in [25]. This will look like a symmetric orbifold of the $n = 2$ CFT's with a marginal deformation turned on. If one is prepared to assume the BPS spectrum is unchanged by the marginal deformation one can argue the requisite bound states arise from the $\mathbb{Z}_{n/2}$ twisted sector of the undeformed orbifold.

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